

# SPECTRUM OF LEBESGUE MEASURE ZERO FOR JACOBI MATRICES OF QUASICRYSTALS

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**ABSTRACT.** We study one-dimensional random Jacobi operators corresponding to strictly ergodic dynamical systems. In this context, we characterize the spectrum of these operators by non-uniformity of the transfer matrices and the set where the Lyapunov exponent vanishes. Adapting this result to subshifts satisfying the so-called Boshernitzan condition, it turns out that the spectrum is supported on a Cantor set with Lebesgue measure zero. This generalizes earlier results for Schrödinger operators.

## 1. INTRODUCTION

Analyzing spectral properties of Schrödinger operators plays an important role in quantum mechanics, where one intends to study the long time behaviour of a particle in a space. The different sorts of potentials are a matter of particular interest in spectral theoretic research areas. Especially, one considers random Schrödinger operators which represent disordered solids. For instance, the periodic model and the Anderson model were intensively examined. In the first case, the potential is completely ordered and periodic. Hence, the corresponding model is appropriate to represent the molecular structure of crystals. In this situation, the spectrum is purely absolutely continuous. By contrast, the potential is absolutely random in the Anderson model, see [And58]. In this context, some results for the spectrum of the corresponding Schrödinger operator are well-known. In detail, the works [FMSS85, CKM87] show that it is purely discrete. Potentials which are aperiodic, i.e. ordered but not periodic, can thematically be classified between these two models. The examination of such potentials has soared up after the year 1982 when Dan Shechtman discovered quasicrystals, see [SBGC84]. First considerations about one-dimensional Schrödinger operators with quasiperiodic potentials can for example be found in [OPR<sup>+</sup>83, KKT83].

From the mathematical point of view, many spectral questions concerning Schrödinger operators with aperiodic potential arise. One issue is to show that the spectrum is a Cantor set of Lebesgue measure zero. If this condition is satisfied we will write (C). In the last decades, two classes of models have attracted special attention in the discrete, one-dimensional case. On the one hand, the class of dynamical systems induced by substitutions has widely been studied. First results about the Fibonacci substitution can be found in [Süt87, Süt89]. In these works, condition (C) is shown under certain conditions. By using trace maps, the absence of point spectrum and (C) is shown for a larger class of primitive substitutions with some reasonable requirements in the works [BBG91, BG93]. In [Dam98], the almost sure absence of eigenvalues was shown for primitive substitutions with the property that the potentials have a local four block structure. On the other hand, the consideration of potentials induced by circle maps has drawn particular attention to itself. Precisely, it is shown in the paper [BIST89] that for irrational numbers, the spectrum is equal to the set where the Lyapunov exponent vanishes. Using this and adapting the Kotani result (cf. [Kot89]) the absence of

absolutely continuous spectrum follows. Further results on upper bounds on the growth of solutions, as well as on the fact that the point spectrum is empty for all Sturmian potentials can be found in [DL99a, DL99b, DKL00].

By using general techniques the work [Len02] proves for a large class of substitutions and Sturmian systems that the corresponding family of Schrödinger operators satisfies (C). In detail, the paper contains a characterization of the spectrum by the Lyapunov exponent and non-uniform transfer matrices. Then (C) holds for a Schrödinger operator induced by a subshift, if the transfer matrices are all uniform. It turns out that the so called Boshernitzan condition, first introduced by Boshernitzan ([Bos85]), for subshifts is suitable to show the uniformity of the transfer matrices, see [DL06a]. Indeed, a large class of models fulfills this condition, such as subshifts satisfying a positive weight condition, all Sturmian subshifts, almost all interval exchange transformations, almost all circle maps and almost all Arnoux-Rauzy subshifts, see [DL06a, DL06b].

The aim of this article is to extend the results of [Len02] to random Jacobi operators arising from a strictly ergodic topological dynamical system  $(\Omega, T)$ . We consider two continuous functions  $p : \Omega \rightarrow \mathbb{R} \setminus \{0\}$  and  $q : \Omega \rightarrow \mathbb{R}$  and its corresponding Jacobi operator on  $\ell^2(\mathbb{Z})$

$$(H_\omega u)(n) := p(T^n \omega) \cdot u(n-1) + p(T^{n+1} \omega) \cdot u(n+1) + q(T^n \omega) \cdot u(n), \quad n \in \mathbb{Z}.$$

This is the discrete version of a Schrödinger operator with weighted Laplacian. It is well-known that there exists a closed subset  $\Sigma \subseteq \mathbb{R}$  such that for all  $\omega \in \Omega$  the equality  $\sigma(H_\omega) = \Sigma$  holds, see e.g. [Len99]. Our purpose is to characterize the spectrum  $\Sigma$  by the non-uniformity of the transfer matrices and the set where the Lyapunov exponent  $\gamma : \mathbb{R} \rightarrow [0, \infty)$  vanishes. In particular, we will verify in our setting that

$$\Sigma = \{E \in \mathbb{R} \mid \gamma(E) = 0\} \bigsqcup \{E \in \mathbb{R} \mid M^E \text{ is not uniform}\}, \quad (\clubsuit)$$

where  $M^E$  is the transfer matrix corresponding to the energy  $E \in \mathbb{R}$ . Applying this result to subshifts and using [Rem11] we will get the following statement: Let  $(\Omega, T)$  be an aperiodic, strictly ergodic dynamical system. If the transfer matrices  $M^E$  are uniform for all energies  $E \in \mathbb{R}$  and if  $p$  and  $q$  take only finitely many values, it follows that  $\Sigma$  fulfills (C), see Theorem 6.4. Since the images of  $p$  and  $q$  are finite sets, we can apply the result of [DL06a] stating that the Boshernitzan condition of a subshift is sufficient for the transfer matrices  $M^E$  being uniform for all energies  $E \in \mathbb{R}$ .

Results of this kind have only been proven for some special cases so far. In detail, in [Yes12] it is shown among other sophisticated results that for Fibonacci sequences with a coupling constant the spectrum is a Cantor set of Lebesgue measure zero. The works [DG08] and [Dah10] prove (C) for Jacobi operators associated with the Fibonacci sequence with vanishing potential  $q$  and positive values of the alphabet. Further elaborations can for instance be found in [JNS09, Mar12].

The paper is organized as follows. In Section 2, we introduce the relevant objects for our model, i.e. the notions of strictly ergodic dynamical systems and of cocycles for continuous, matrix-valued functions. Next, we examine these objects in more detail in Section 3. This includes growth behaviour and continuity properties. Section 4 is devoted to random Jacobi operators induced by a strictly ergodic dynamical system. In this context, we prove important connections between the Lyapunov exponent of the corresponding transfer matrices and the spectral properties of the operator. These considerations lead to our main result in Section

5, where we give a complete description of the spectrum in terms of the Lyapunov exponent and the uniformity property of the transfer matrices. Precisely, we show the Equality ( $\clubsuit$ ), cf. Theorem 5.1. Finally, we apply our results to subshifts in Section 6. In fact, we show in Theorem 6.4 that for a very large class of operator families, uniformity of all transfer matrices implies (C).

## 2. GENERALITIES

We start by defining the relevant objects for our work. To do so, we introduce the notion of cocycles induced by some measure preserving, strictly ergodic transformation on a topological probability space. Further, we apply a version of Kingman's subadditive ergodic theorem in order to get some almost-sure approximation results for the growth rate of the underlying matrix norms.

Let  $(\Omega, T)$  be a dynamical system, where  $\Omega$  is a compact metric space and  $T : \Omega \rightarrow \Omega$  is a homeomorphism. Let  $\mu$  be a probability measure on the Borel  $\sigma$ -algebra on  $\Omega$ . The measure  $\mu$  is called *invariant*, if for all  $A \in \mathcal{F}$

$$\mu(T(A)) = \mu(A).$$

The dynamical system is called *ergodic*, if each measurable  $A$  with  $A = T^{-1}(A)$  has measure one or zero. A dynamical system is called *uniquely ergodic* if there exists only one invariant ergodic probability measure on  $\mathcal{F}$ . Further, it is called *minimal*, if every orbit is dense in  $\Omega$ . If  $(\Omega, T)$  is both uniquely ergodic and minimal, it is called *strictly ergodic*.

Consider the general linear group  $GL(2, \mathbb{R})$  of 2x2 matrices with real values and nonzero determinant and the special linear group  $SL(2, \mathbb{R})$  as the subgroup of those matrices with determinant one. The topology on these groups is defined by the operator norm  $\|\cdot\|$ . For a continuous map  $M : \Omega \rightarrow GL(2, \mathbb{R})$  we define the cocycle  $M(n, \omega)$  for  $\omega \in \Omega$  and  $n \in \mathbb{Z}$  by

$$M(n, \omega) := \begin{cases} M(T^{n-1}\omega) \cdot \dots \cdot M(\omega) & : n > 0 \\ Id & : n = 0 \\ M^{-1}(T^n\omega) \cdot \dots \cdot M^{-1}(T^{-1}\omega) & : n < 0. \end{cases}$$

Note that the equality

$$M(m, T^n\omega) \cdot M(n, \omega) = M(m+n, \omega)$$

holds for each  $m, n \in \mathbb{Z}$  and  $\omega \in \Omega$ . The following proposition states a well-known version of Kingman's subadditive ergodic theorem, see e.g. [KW82].

**Proposition 2.1.** *Let  $(\Omega, T)$  be uniquely ergodic with invariant probability measure  $\mu$  and  $M : \Omega \rightarrow GL(2, \mathbb{R})$  be continuous. Then for*

$$\Lambda(M) := \inf_{n \in \mathbb{N}} \frac{1}{n} \cdot \int_{\Omega} \log \|M(n, \omega)\| d\mu(\omega)$$

*the equality*

$$\Lambda(M) = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \log \|M(n, \omega)\|$$

*holds for  $\mu$ -a.e.  $\omega \in \Omega$ .*

Following in [Fur97], we use the following definition. It is motivated by the fact that unique ergodicity of  $(\Omega, T)$  is equivalent to the uniform convergence of the ergodic averages in the case of continuous functions.

**Definition 2.2.** Let  $(\Omega, T)$  be strictly ergodic. A continuous map  $M : \Omega \rightarrow GL(2, \mathbb{R})$  is called *uniform*, if the limit

$$\Lambda(M) = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \log \|M(n, \omega)\|$$

exists for all  $\omega \in \Omega$  and converges uniformly in  $\omega \in \Omega$ .

Note that in the paper of [Fur97], it is not required that the dynamical system  $(\Omega, T)$  is minimal. However, it is convenient for our setting to assume minimality. For minimal topological dynamical systems, uniform existence of the limit implies uniform convergence as shown by Weiss, cf. [Len04] as well.

### 3. KEY RESULTS

In this section, we provide general facts for strictly ergodic cocycles  $M : \Omega \rightarrow SL(2, \mathbb{R})$ . They will be exploited in the proofs of the spectral theoretic statements for Jacobi operators in the following Sections 4 and 5.

The next assertion can be found for example in [Len02], Lemma 3.2.

**Lemma 3.1.** *Let  $(\Omega, T)$  be a dynamical system which is strictly ergodic with invariant probability measure  $\mu$ . Consider a uniform  $M : \Omega \rightarrow SL(2, \mathbb{R})$  with  $\Lambda(M) > 0$ . Then for each  $u \in \mathbb{R}^2 \setminus \{0\}$  and  $\omega \in \Omega$  there are constants  $D, \kappa > 0$  such that*

$$\|M(n, \omega)u\| \geq D \cdot e^{\kappa \cdot |n|}$$

for all  $n \geq 0$  or  $n \leq 0$ .

The following notion is inspired by [Fur97] and [Len04]. Let  $\mathcal{U}(\Omega)$  be the set of uniform, continuous maps  $M : \Omega \rightarrow SL(2, \mathbb{R})$ . Further, the set  $\mathcal{U}(\Omega)_+$  are the elements  $M \in \mathcal{U}(\Omega)$  where  $\Lambda(M) > 0$ . A metric on the complete metric space of continuous maps defined on  $\Omega$  with values in  $SL(2, \mathbb{R})$  is given by

$$d(A, B) := \sup_{\omega \in \Omega} \|A(\omega) - B(\omega)\|,$$

see [Fur97]. Denote this complete metric space by  $\mathcal{C}(\Omega, SL(2, \mathbb{R}))$ .

The proof of Theorem 3.2 can be found in [Fur97], cf. [Len04] as well.

**Theorem 3.2.** *Let  $(\Omega, T)$  be strictly ergodic. Then the set  $\mathcal{U}(\Omega)_+$  is open in the space  $\mathcal{C}(\Omega, SL(2, \mathbb{R}))$  and the map  $\Lambda : \mathcal{U}(\Omega) \rightarrow \mathbb{R}$  is continuous.*

The next assertion is an adaption of a result of [Fur97], see [Len02] as well.

**Lemma 3.3.** *Let  $(\Omega, T)$  be strictly ergodic. Consider a uniform  $M : \Omega \rightarrow SL(2, \mathbb{R})$  and a sequence of continuous maps  $M_n : \Omega \rightarrow SL(2, \mathbb{R})$  where  $d(M_n, M)$  tends to zero. Then*

$$\Lambda(M_n) \xrightarrow{n \rightarrow \infty} \Lambda(M).$$

*Proof.* As a consequence of Theorem 3.2 the convergence of  $\Lambda(M_n)$  to  $\Lambda(M)$  follows if  $d(M_n, M)$  and  $d(M_n^{-1}, M^{-1})$  converge to zero. If  $\lim_{n \rightarrow \infty} d(M_n, M) = 0$  a short computation leads to  $\lim_{n \rightarrow \infty} d(M_n^{-1}, M^{-1}) = 0$ . ■

The next statements provide useful tools to show the uniformity in some situations which are convenient for our purpose. Lemma 3.4 provides an upper bound for the logarithmic growth of the norm of a continuous map  $M : \Omega \rightarrow GL(2, \mathbb{R})$ , see [Fur97], Corollary 2.

**Lemma 3.4.** *Let  $(\Omega, T)$  be uniquely ergodic and consider a continuous  $M : \Omega \rightarrow GL(2, \mathbb{R})$ . Then*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \cdot \log \|M(n, \omega)\| \leq \Lambda(M)$$

*uniformly on  $\Omega$ .*

**Lemma 3.5.** *Consider a uniquely ergodic dynamical system  $(\Omega, T)$ . Let  $M : \Omega \rightarrow GL(2, \mathbb{R})$  and  $\widetilde{M} : \Omega \rightarrow GL(2, \mathbb{R})$  be continuous. If there is a constant  $K \geq 1$  independent of  $\omega \in \Omega$  and  $n \in \mathbb{Z}$  such that*

$$\|M(n, \omega)\| \leq K \cdot \|\widetilde{M}(n, \omega)\| \text{ and } \|\widetilde{M}(n, \omega)\| \leq K \cdot \|M(n, \omega)\|,$$

*then the following two statements hold.*

- (i) *The equality  $\Lambda(M) = \Lambda(\widetilde{M})$  holds.*
- (ii) *The map  $M$  is uniform, if and only if  $\widetilde{M}$  is uniform as well.*

*Proof.* This proof is straight forward by a short computation. ■

**Lemma 3.6.** *Let  $M : \Omega \rightarrow GL(2, \mathbb{R})$  be continuous such that*

$$M(\omega) = C^{-1}(T\omega) \cdot \begin{pmatrix} f_1(\omega) & 0 \\ 0 & f_2(\omega) \end{pmatrix} \cdot C(\omega)$$

*where  $|f_1|, |f_2| \in \mathcal{C}(\Omega, \mathbb{R})$  and  $C : \Omega \rightarrow GL(2, \mathbb{R})$  is such that  $\|C\|, \|C^{-1}\| : \Omega \rightarrow \mathbb{R}$  are continuous. Then the function  $M$  is uniform.*

*Proof.* First note that for  $\omega \in \Omega$  the equality

$$M(n, \omega) = C^{-1}(T^n \omega) \cdot \underbrace{\begin{pmatrix} \prod_{j=0}^n f_1(T^j \omega) & 0 \\ 0 & \prod_{j=0}^n f_2(T^j \omega) \end{pmatrix}}_{=: A(n, \omega)} \cdot C(\omega), \quad n \in \mathbb{N}$$

holds. Since  $\|C\|, \|C^{-1}\| : \Omega \rightarrow \mathbb{R}$  are continuous and  $\Omega$  is compact, we immediately get that there are constants  $C_1, C_2 > 0$  such that  $C_1 \cdot \|A(n, \omega)\| \leq \|M(n, \omega)\| \leq C_2 \cdot \|A(n, \omega)\|$  for all  $n \in \mathbb{N}$  and every  $\omega \in \Omega$ .

Moreover, we know that all norms defined on the linear space of matrices are equivalent and so, there are constants  $D_1, D_2 > 0$  such that

$$D_1 \cdot \max \left\{ \prod_{j=0}^n |f_1(T^j \omega)|, \prod_{j=0}^n |f_2(T^j \omega)| \right\} \leq \|M(n, \omega)\| \leq D_2 \cdot \max \left\{ \prod_{j=0}^n |f_1(T^j \omega)|, \prod_{j=0}^n |f_2(T^j \omega)| \right\}$$

Since  $M$  is invertible it follows that  $f_1$  and  $f_2$  never vanish and so, the functions  $\log \circ |f_1|$  and  $\log \circ |f_2|$  are well-defined and continuous by our requirements. Using unique ergodicity, it follows from standard arguments in ergodic theory (see e.g. [EFHN09], Corollary 9.9) that

$\frac{1}{n} \sum_{j=0}^n \log \circ |f_1| \circ T^j$  and  $\frac{1}{n} \sum_{j=0}^n \log \circ |f_2| \circ T^j$  converge uniformly on  $\Omega$  to some constant. Hence,  $\frac{1}{n} \log \|M(n, \omega)\|$  converge uniformly on  $\Omega$  to a constant.  $\blacksquare$

#### 4. JACOBI MATRICES

We present the notion of a uniquely ergodic family  $\{H_\omega\}_{\omega \in \Omega}$  of Jacobi operators acting on  $\ell^2(\mathbb{Z})$ . For each  $E \in \mathbb{R}$ , we define the transfer matrices  $M^E$  corresponding to the equation  $(H_\omega - E)u = 0$  as cocycle functions on  $\Omega$ . Further, we verify the existence of the Lyapunov exponent  $\gamma(E)$  containing information on the growth rate of the matrix norms, cf. Lemma 4.1. By drawing connections between the Lyapunov exponent and the spectrum of the operators, we prove major preparations for the main results of this work (cf. Section 5) in the Proposition 4.4, Lemmas 4.7 and 4.10.

We consider two continuous maps  $p : \Omega \rightarrow \mathbb{R} \setminus \{0\}$  and  $q : \Omega \rightarrow \mathbb{R}$  and for  $\omega \in \Omega$  its corresponding Jacobi operator  $H_\omega : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$  defined by

$$(H_\omega u)(n) := p(T^{n+1}\omega) \cdot u(n+1) + q(T^n\omega) \cdot u(n) + p(T^n\omega) \cdot u(n-1).$$

Denote  $p(T^n\omega)$  by  $a_\omega(n)$  and  $q(T^n\omega)$  by  $b_\omega(n)$ . Since  $\Omega$  is compact and  $p$  and  $q$  are continuous there is a constant  $K \geq 1$  such that

$$\frac{1}{K} \leq |a_\omega(\cdot)| \leq K$$

and

$$0 \leq |b_\omega(\cdot)| \leq K$$

for each  $\omega \in \Omega$ . Using this boundeness it follows that the norm of  $H_\omega$  is bounded, because

$$\|H_\omega\| \leq 2 \cdot \|a_\omega\|_\infty + \|b_\omega\|_\infty \leq 3 \cdot K.$$

For  $\omega \in \Omega$  and  $E \in \mathbb{R}$ , we are interested in general solutions of the difference equation

$$a_\omega(n+1) \cdot u(n+1) + b_\omega(n) \cdot u(n) + a_\omega(n) \cdot u(n-1) - E \cdot u(n) = 0 \quad (\spadesuit)$$

for  $n \in \mathbb{Z}$ . We define the so called *transfer matrix* by

$$M^E(\omega) := \begin{pmatrix} \frac{E-b_\omega(1)}{a_\omega(2)} & -\frac{a_\omega(1)}{a_\omega(2)} \\ 1 & 0 \end{pmatrix}.$$

Similarly to the Schrödinger case it follows that  $(\spadesuit)$  holds, if and only if

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = M^E(n, \omega) \cdot \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}$$

for all  $n \in \mathbb{Z}$ . Unlike to the classical case of Schrödinger operators the determinant of  $M^E(\omega)$  is not necessarily equal to one. Thus, we introduce the following matrix

$$\widetilde{M}^E(\omega) := \begin{pmatrix} \frac{E-b_\omega(1)}{a_\omega(2)} & -\frac{1}{a_\omega(2)} \\ a_\omega(2) & 0 \end{pmatrix}$$

with determinant equal to one. Then the equation  $(\spadesuit)$  holds, if and only if

$$\begin{pmatrix} u(n+1) \\ a_\omega(n+1) \cdot u(n) \end{pmatrix} = \widetilde{M}^E(n, \omega) \cdot \begin{pmatrix} u(1) \\ a_\omega(1) \cdot u(0) \end{pmatrix}.$$

Note that the maps  $M^E$  and  $\widetilde{M}^E$  are continuous by definition.

**Lemma 4.1.** *Let  $(\Omega, T)$  be strictly ergodic and consider the maps  $M^E : \Omega \rightarrow GL(2, \mathbb{R})$  and  $\widetilde{M}^E : \Omega \rightarrow SL(2, \mathbb{R})$  defined as above. Then for an  $\omega \in \Omega$  the limit*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|M^E(n, \omega)\|$$

*exists if and only if the limit*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\widetilde{M}^E(n, \omega)\|$$

*exists and in these cases, they are equal. Moreover,  $M^E$  is uniform if and only if  $\widetilde{M}^E$  is uniform.*

*Proof.* Define the continuous map  $C : \Omega \rightarrow GL(2, \mathbb{R})$  by

$$C(\omega) := \begin{pmatrix} 1 & 0 \\ 0 & a_\omega(1) \end{pmatrix}.$$

Then we get the equation  $M^E(\omega) = C^{-1}(\omega) \widetilde{M}^E(\omega) C(\omega)$ . This yields to the equality  $M^E(n, \omega) = C^{-1}(T^n \omega) \widetilde{M}^E(n, \omega) C(\omega)$ . By Lemma 3.5 our statements follows.  $\blacksquare$

According to Proposition 2.1 we can define the *Lyapunov exponent* for the energy  $E \in \mathbb{R}$  by  $\gamma(E) := \Lambda(M^E)$ .

**Lemma 4.2.** *The Lyapunov exponent  $\gamma(E)$  is greater or equal than zero for all  $E \in \mathbb{R}$ .*

*Proof.* Since  $\det(\widetilde{M}^E(T^n \omega)) = 1$  for each  $n \in \mathbb{N}$  it follows that  $\det(\widetilde{M}^E(n, \omega)) = 1$  for all  $n \in \mathbb{N}$ . Thus, the norm  $\|\widetilde{M}^E(n, \omega)\|$  is greater or equal than one and so

$$\gamma(E) \stackrel{\text{L. 4.1}}{=} \Lambda(\widetilde{M}^E) \stackrel{\text{L. 3.4}}{\geq} \limsup_{n \rightarrow \infty} \frac{1}{n} \cdot \log(\underbrace{\|\widetilde{M}^E(n, \omega)\|}_{\geq 1}) \geq 0.$$

$\blacksquare$

The following well-known Proposition 4.3 states that the spectrum of the Jacobi operators with respect to the elements of  $\Omega$  does not change, if the dynamical system  $(\Omega, T)$  is minimal. Hence, one also can talk about the spectrum of the whole family of Jacobi operators, see e.g. [Len99].

**Proposition 4.3.** *Let  $(\Omega, T)$  be a minimal dynamical system. Then there exists a set  $\Sigma \subseteq \mathbb{R}$  such that  $\sigma(H_\omega) = \Sigma$  for every  $\omega \in \Omega$ .*

As a direct consequence of Lemma 3.1 and some general result, Proposition 4.4 states a sufficient condition for  $E \in \mathbb{R}$  not to be contained in the spectrum of a family of Jacobi operators.

**Proposition 4.4.** *Let  $(\Omega, T)$  be strictly ergodic and  $p : \Omega \rightarrow \mathbb{R} \setminus \{0\}$  and  $q : \Omega \rightarrow \mathbb{R}$  be continuous maps with corresponding family of Jacobi operators  $(H_\omega)_{\omega \in \Omega}$ . If for  $E \in \mathbb{R}$  we have  $\Lambda(M^E) > 0$  and  $M^E$  is uniform, then  $E$  does not belong to the spectrum  $\Sigma$  of the family of Jacobi operators  $(H_\omega)_{\omega \in \Omega}$ .*

*Proof.* By Lemma 4.1  $\widetilde{M}^E$  is uniform and  $\Lambda(\widetilde{M}^E) > 0$ . Recall the definition of the metric  $d$  on  $\mathcal{C}(\Omega, SL(2, \mathbb{R}))$  defined in the beginning of Section 3. Note that with respect to this metric, for each  $\varepsilon > 0$  there exists an interval  $I(\varepsilon) \ni E$  such that for all  $F \in I(\varepsilon)$  we have  $d(\widetilde{M}^F, \widetilde{M}^E) < \varepsilon$ . According to Theorem 3.2 the set

$$\mathcal{U}(\Omega)_+ := \{M : \Omega \rightarrow SL(2, \mathbb{R}) \mid M \text{ continuous, uniform and } \Lambda(M) > 0\}$$

is open. Thus, there exists an open interval  $I$  containing  $E$  such that for all  $F \in I$  we have  $M^F \in \mathcal{U}(\Omega)_+$ .

Choose one  $\omega \in \Omega$  and assume the contrary i.e.  $E \in \Sigma$ . Thus, there exists spectrum of  $H_\omega$  in  $I$ . Consequently, the spectral measure of  $H_\omega$  gives actually weight to  $I$ . Hence, there must be a solution  $(u(n))_{n \in \mathbb{Z}}$  of the difference equation  $(\spadesuit)$  for one  $F \in I$  which is polynomially bounded and not zero, see [CL90], Theorem II.4.5. Since  $\widetilde{M}^F \in \mathcal{U}(\Omega)_+$  and  $(u(n))_{n \in \mathbb{Z}}$  solves  $(\spadesuit)$  there are constants  $K \geq 1$  and  $\kappa, D > 0$  such that

$$K \cdot \left\| \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} \right\| \geq \left\| \begin{pmatrix} u(n+1) \\ a_\omega(n+1) \cdot u(n) \end{pmatrix} \right\| = \left\| \widetilde{M}^F(n, \omega) \begin{pmatrix} u(1) \\ a_\omega(1) \cdot u(0) \end{pmatrix} \right\| \geq D \cdot e^{\kappa \cdot |n|}$$

for  $n \geq 0$  or  $n \leq 0$ , see Lemma 3.1. This contradicts the polynomial boundedness of  $(u(n))_{n \in \mathbb{Z}}$ .  $\blacksquare$

The next step will be to prove a general result that the set where the Lyapunov exponent vanishes is contained in the spectrum under certain conditions to the dynamical system. In order to do so, we use the notion of a subexponentially increasing sequence. In detail,  $(u(n))_{n \in \mathbb{Z}}$  is called *subexponentially increasing*, if

$$\limsup_{|n| \rightarrow \infty} \frac{1}{|n|} \log |u(n)| \leq 0.$$

By some elementary arguments, one can check that this is equivalent to the fact that

$$\limsup_{|n| \rightarrow \infty} \frac{1}{|n|} \log \left( \sum_{j=-|n|}^{|n|} |u(j)|^2 \right)^{\frac{1}{2}} \leq 0.$$

Define for  $(u(n))_{n \in \mathbb{Z}}$  a new sequence

$$u_l(k) := \mathbf{1}_{[-l, l]}(k) \cdot u(k), \quad k \in \mathbb{Z}$$

where  $\mathbf{1}_{[-l, l]}(k)$  is equal to one if  $k \in [-l, l]$  and zero else.

**Lemma 4.5.** *Let  $(u(n))_{n \in \mathbb{Z}} \subseteq \mathbb{R}$  be a subexponentially increasing sequence. Then for all  $\delta > 0$  there is some  $n(\delta) \in \mathbb{N}$  such that for each  $l \in \mathbb{N}$  with  $l \geq n(\delta)$  we have*

$$\|u_{l+1}\|^2 \leq e^\delta \cdot \|u_{l-1}\|^2$$

*Proof.* Assume the contrary, which means that for every  $\delta > 0$  and  $l(\delta) \in \mathbb{N}$  there is an  $l \geq l(\delta)$  such that  $\|u_{l+1}\|^2 > e^\delta \|u_{l-1}\|^2$ . Without loss of generality, consider the subsequence  $(l_k)_{k \in \mathbb{N}}$  such that  $l_k$  is even for each  $k \in \mathbb{N}$  and such that the latter inequality holds. Then  $\|u_{l_k+1}\| > e^{\delta \cdot \frac{l_k}{2}} \cdot \|u_1\|$  and so,

$$\frac{1}{l_k+1} \cdot \log \|u_{l_k+1}\| > \frac{\delta}{2} \cdot \underbrace{\frac{l_k}{l_k+1}}_{\geq \frac{1}{2}} + \frac{1}{l_k+1} \cdot \log \|u_1\| \geq \frac{\delta}{4} + \frac{1}{l_k+1} \log \|u_1\|.$$



Hence,

$$\limsup_{k \rightarrow \infty} \frac{1}{l_k + 1} \log \|u_{l_k+1}\| \geq \liminf_{k \rightarrow \infty} \frac{1}{l_k + 1} \log \|u_{l_k+1}\| > \frac{\delta}{4} > 0$$

contradicting the subexponential growth.  $\blacksquare$

The following well-known statement can for example be found in [CL90], Proposition V.4.1. We give the proof for the sake of completeness.

**Lemma 4.6.** *Let  $(\Omega, T)$  be strictly ergodic. Then*

$$\Gamma := \{E \in \mathbb{R} \mid \gamma(E) = 0\} \subseteq \Sigma.$$

*Proof.* Consider some  $E \in \Gamma$  and choose one  $\underline{u} \in \mathbb{R}^2$  as initial condition with  $\|\underline{u}\| = 1$ . Let  $(u(n))_{n \in \mathbb{Z}}$  be a solution of the difference equation  $(\spadesuit)$  for some  $\omega \in \Omega$  with  $\begin{pmatrix} u(1) \\ u(0) \end{pmatrix} := \underline{u}$ .

Then Lemma 3.5 and a short computation lead to  $\Lambda(M^E) = \Lambda((M^E)^{-1})$ . Consequently, the inequality

$$\limsup_{|n| \rightarrow \infty} \frac{1}{|n|} \log \left\| \begin{pmatrix} u(n) \\ u(n+1) \end{pmatrix} \right\| \leq 0$$

follows by Lemma 3.4 meaning that  $(u(n))_{n \in \mathbb{Z}}$  is subexponentially increasing.

Recapitulate the notion of  $u_l(n) := \mathbf{1}_{[-l, l]}(n) \cdot u(n)$  ( $n \in \mathbb{Z}$ ) for  $l \in \mathbb{N}$ . Then by using the subexponential growth of  $(u(n))_{n \in \mathbb{Z}}$  there is for all  $\delta > 0$  some  $l(\delta) \in \mathbb{N}$  such that for each  $l \in \mathbb{N}$  with  $l \geq l(\delta)$  it follows

$$\begin{aligned} \|(H_\omega - E)u_l\|^2 &\stackrel{C := \|H_\omega\| + |E|}{\leq} C \cdot \left( \underbrace{\|u_{l+1}\|^2}_{\leq e^\delta \|u_{l-1}\|^2, \text{L. 4.5}} - \|u_{l-1}\|^2 \right) \\ &\leq C(e^\delta - 1) \cdot \|u_l\|^2. \end{aligned}$$

Since the expression  $(e^\delta - 1)$  converges to zero as  $\delta$  tends to zero we can choose a diagonal subsequence  $(l_k)_{k \in \mathbb{N}}$  such that

$$\left\| (H_\omega - E) \frac{u_{l_k}}{\|u_{l_k}\|} \right\| \xrightarrow{k \rightarrow \infty} 0.$$

Thus, we have constructed a Weyl sequence for  $E \in \Gamma$  with respect to the operator  $H_\omega$ . Hence, by general results, it follows that  $E$  is an element of the spectrum  $\Sigma$ .  $\blacksquare$

Proposition 4.4 and Lemma 4.6 yield to the following statement.

**Lemma 4.7.** *Let  $(\Omega, T)$  be strictly ergodic. If  $M^E$  is uniform for every  $E \in \mathbb{R}$ , then  $\Sigma = \Gamma$  and the Lyapunov exponent  $\gamma : \mathbb{R} \rightarrow [0, \infty)$  is continuous.*

*Proof.* The equation  $\Sigma = \Gamma$  is a direct consequence of Proposition 4.4 and Lemma 4.6. A short calculation using Lemma 3.3 shows that  $\Lambda(\widetilde{M}^E)$  is continuous. By Lemma 4.1, this holds also true for  $\gamma$ .  $\blacksquare$

**Lemma 4.8.** *Let  $(\Omega, T)$  be strictly ergodic. For  $E \in \mathbb{R}$  with  $\gamma(E) = 0$  it follows that  $M^E$  is uniform.*

*Proof.* This follows immediately from Lemma 3.4, Lemma 4.1 and Lemma 4.2.  $\blacksquare$

The following statement, well-known under the name Combes/Thomas argument, can be proven along the lines of [Kir08] (Theorem 11.2) by adjusting constants. For the convenience of the reader, we give a sketch of the proof.

**Proposition 4.9.** *Let  $(\Omega, T)$  be a dynamical system and let  $(H_\omega)_{\omega \in \Omega}$  be the family of the corresponding Jacobi operators, defined as above. Let  $K \geq 1$  be the constant such that  $\frac{1}{K} \leq |a_\omega(\cdot)| \leq K$  and  $0 \leq |b_\omega(\cdot)| \leq K$ . For  $\omega \in \Omega$  and some  $E \in \mathbb{R} \setminus \sigma(H_\omega)$  set  $\eta := \text{dist}(E, \sigma(H_\omega)) > 0$ , then for each  $n, m \in \mathbb{Z}$  there exists a constant  $\kappa := \kappa(\eta) > 0$  such that the inequality*

$$|\langle \delta_n \mid (H_\omega - E)^{-1} \delta_m \rangle| \leq \frac{2}{\eta} \cdot e^{-\kappa|n-m|}$$

holds where  $\delta_k(k) = 1$  and  $\delta_k(n) = 0$  for  $n \neq k$ .

*Proof.* Let  $\kappa := \frac{\eta}{K} \cdot c$ , where  $c > 0$  is some constant such that  $2ce^\kappa \leq \frac{1}{2}$  and fix an arbitrary  $\omega \in \Omega$ . For  $k \in \mathbb{Z}$  define the multiplication operator  $\mathcal{M}_k : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$  by

$$\mathcal{M}_k u(n) := e^{\kappa \cdot |k-n|} \cdot u(n), \quad n \in \mathbb{Z}.$$

By using for some operator  $A$  on  $\ell^2(\mathbb{Z})$  the equality

$$\langle \delta_n \mid (\mathcal{M}_k^{-1} A \mathcal{M}_k) \delta_m \rangle = e^{-\kappa \cdot |k-n|} \cdot \langle \delta_n \mid A \delta_m \rangle \cdot e^{\kappa \cdot |k-m|}, \quad n, m \in \mathbb{Z}$$

it follows

$$|\langle \delta_n \mid (H_\omega - E)^{-1} \delta_m \rangle| \leq e^{-\kappa \cdot |n-m|} \cdot \left\| (\mathcal{M}_k^{-1} H_\omega \mathcal{M}_k - E)^{-1} \right\|, \quad n, m \in \mathbb{Z}.$$

Applying the resolvent equation we get

$$(\mathcal{M}_k^{-1} H_\omega \mathcal{M}_k - E)^{-1} \cdot (1 + (\mathcal{M}_k^{-1} H_\omega \mathcal{M}_k - H_\omega) \cdot (H_\omega - E)^{-1}) = (H_\omega - E)^{-1}.$$

Recall that  $|a_\omega(\cdot)| \leq K$  and  $|b_\omega(\cdot)| \leq K$  for some constant  $K \geq 1$  which implies that  $|\langle \delta_n \mid H_\omega \delta_m \rangle| \leq K$  for  $n, m \in \mathbb{Z}$ . We will invert  $(1 + (\mathcal{M}_k^{-1} H_\omega \mathcal{M}_k - H_\omega) \cdot (H_\omega - E)^{-1})$  by using the von Neumann series. In order to do so, we have to check that the norm of

$$(\mathcal{M}_k^{-1} H_\omega \mathcal{M}_k - H_\omega) \cdot (H_\omega - E)^{-1}$$

is smaller than one. For  $n \in \mathbb{Z}$  we get by a short computation that

$$\begin{aligned} \sum_{m \in \mathbb{Z}} |\langle \delta_n \mid (\mathcal{M}_k^{-1} H_\omega \mathcal{M}_k - H_\omega) \delta_m \rangle| &\leq \sum_{\substack{m \in \mathbb{Z} \\ |m-n|=1}} \left| e^{\kappa \cdot (|k-m| - |k-n|)} - 1 \right| \cdot |\langle \delta_n \mid H_\omega \delta_m \rangle| \\ &\leq \sum_{\substack{m \in \mathbb{Z} \\ |m-n|=1}} \kappa e^\mu K \\ &\leq 2K \mu e^\mu. \end{aligned}$$

Consequently,

$$\|\mathcal{M}_k^{-1} H_\omega \mathcal{M}_k - H_\omega\| \leq 2K \kappa e^\kappa$$

which leads to

$$\left\| (\mathcal{M}_k^{-1} H_\omega \mathcal{M}_k - H_\omega) \cdot (H_\omega - E)^{-1} \right\| \leq 2K \kappa e^\kappa \cdot \frac{1}{\eta} = 2ce^\kappa \leq \frac{1}{2}.$$

Hence, by the norm estimate for von Neumann series

$$\left\| \left( 1 + (\mathcal{M}_k^{-1} H_\omega \mathcal{M}_k - H_\omega) \cdot (H_\omega - E)^{-1} \right)^{-1} \right\| \leq 2$$

and by the previous considerations

$$(\mathcal{M}_k^{-1} H_\omega \mathcal{M}_k - E)^{-1} = (H_\omega - E)^{-1} \left( 1 + (\mathcal{M}_k^{-1} H_\omega \mathcal{M}_k - H_\omega) \cdot (H_\omega - E)^{-1} \right)^{-1}.$$

By the definition of  $\kappa$  and the fact that  $\|(H_\omega - E)^{-1}\| \leq \frac{1}{\eta}$  this implies

$$|\langle \delta_n \mid (H_\omega - E)^{-1} \delta_m \rangle| \leq e^{-\kappa \cdot |n-m|} \cdot \left\| (\mathcal{M}_k^{-1} H_\omega \mathcal{M}_k - E)^{-1} \right\| \leq \frac{2}{\eta} \cdot e^{-\kappa \cdot |n-m|}.$$

■

The next statement follows the lines of [Len02], Lemma 4.3 and [Len04], Theorem 3.

**Lemma 4.10.** *Let  $(\Omega, T)$  be strictly ergodic and  $E \in \mathbb{R} \setminus \Sigma$ . Then  $M^E$  is uniform and  $\gamma(E) > 0$ .*

*Proof.* Let  $E \in \mathbb{R} \setminus \Sigma = \mathbb{R} \setminus \sigma(H_\omega)$  for  $\omega \in \Omega$ . In Lemma 4.6 it is shown that  $\Gamma \subseteq \Sigma$  is a general result and so,  $\gamma(E) > 0$  for  $E \in \mathbb{R} \setminus \Sigma$ . According to Lemma 4.1 we have  $\Lambda(\widetilde{M}^E) > 0$ .

In the following, we will first show for  $\omega \in \Omega$  that there exist two unique (up to a sign) normalized vectors  $u(\omega)$  and  $v(\omega)$  such that they satisfy the following condition. The norm  $\|\widetilde{M}^E(n, \omega) u(\omega)\|$  decays exponentially, if  $n$  tends to  $\infty$  and similarly,  $\|\widetilde{M}^E(-n, \omega) v(\omega)\|$  decays exponentially, if  $n$  goes to  $\infty$ . Secondly, we will use these normalized vectors to construct a diagonalization of  $\widetilde{M}^E$  as in Lemma 3.6 leading to the uniformity of  $M^E$ .

For  $\omega \in \Omega$ , set

$$u_i(n) := (H_\omega - E)^{-1} \delta_i(n) = \langle \delta_n \mid (H_\omega - E)^{-1} \delta_i \rangle, \quad n \in \mathbb{Z},$$

for  $i \in \mathbb{Z}$  which is an element of  $\ell^2(\mathbb{Z})$ . Fix one  $\omega \in \Omega$  and consider the vectors

$$\underline{u}_0 := \begin{pmatrix} u_0(0) \\ u_0(1) \end{pmatrix} \quad \underline{u}_{-1} := \begin{pmatrix} u_{-1}(0) \\ u_{-1}(1) \end{pmatrix}.$$

Note that for  $u \in \ell^2(\mathbb{Z})$  and  $n \in \mathbb{Z}$  the value  $(H_\omega u)(n)$  depends only on  $u(n-1)$ ,  $u(n)$  and  $u(n+1)$ . Further,  $(H_\omega - E) u_0(0) = 1$  and  $|u_0(-1)| = |u_{-1}(0)|$ . Consequently, at least one of the numbers  $u_0(0)$ ,  $u_0(1)$  and  $u_{-1}(0)$  is not zero. Thus, we can normalize one of the vectors  $\underline{u}_0$  and  $\underline{u}_{-1}$ . Without loss of generality, let  $\underline{u}_0$  be the vector which can be normalized. Denote its normalized vector by  $u(\omega) := \frac{1}{\|\underline{u}_0\|} \cdot \underline{u}_0$ . By definition we have  $(H_\omega - E) u_0(n) = 0$  for  $n \geq 1$  implying that  $u_0$  is a solution to the right. Hence,

$$\widetilde{M}^E(n, \omega) u(\omega) = \frac{1}{\|\underline{u}_0\|} \cdot \begin{pmatrix} u_0(n) \\ a_\omega(n+1) \cdot u_0(n+1) \end{pmatrix}$$

and so,

$$\|\widetilde{M}^E(n, \omega) u(\omega)\| \leq \widetilde{K} \cdot (|u_0(n)| + |u_0(n+1)|), \quad n \geq 1$$

where  $\widetilde{K} > 0$  is some constant independent of  $n \geq 1$ . In accordance with Proposition 4.9, for  $i \in \mathbb{Z}$  there exist a constant  $D(i)$  and  $\kappa(i)$  such that  $|u_i(n)| \leq D(i) \cdot \exp(-\kappa(i) \cdot |n|)$  for  $n \in \mathbb{Z}$ . Thus, by Proposition 4.9 we get that  $\|\widetilde{M}^E(n, \omega) u(\omega)\|$  decays exponentially, if  $n$  goes

to  $\infty$ . Analogously, we construct  $v(\omega) := \frac{1}{\|u_i\|} \cdot \underline{u}_i$  for  $i$  equal to 1 or 2. As above, we get that  $\|\widetilde{M}^E(-n, \omega) v(\omega)\|$  decays exponentially, if  $n$  tends to  $\infty$ .

Since, for all  $i \in \mathbb{Z}$ , the map  $\omega \mapsto (H_\omega - E)^{-1} \delta_i$  is continuous with respect to the topology on  $\Omega$  it follows that  $u(\omega)$  and  $v(\omega)$  can be chosen in a continuous dependency on  $\Omega$ . Now we will verify that these vectors are unique up to a sign and linearly independent.

Assume that  $u(\omega)$  and  $v(\omega)$  are not linearly independent meaning that there is some  $0 \neq c \in \mathbb{R}$  such that  $u(\omega) = c \cdot v(\omega)$ . Let  $(\alpha(n))_{n \in \mathbb{Z}}$  be the sequence satisfying  $\begin{pmatrix} \alpha(n+1) \\ \alpha(n) \end{pmatrix} = \widetilde{M}^E(n, \omega) \cdot u(\omega)$  for  $n \in \mathbb{Z}$ , which is a solution of the difference equation  $(\spadesuit)$ . Moreover, by the above consideration

$$\|\widetilde{M}^E(n, \omega) u(\omega)\| \leq D e^{-\kappa \cdot n}, \quad n \in \mathbb{N}$$

and

$$\|\widetilde{M}^E(-n, \omega) u(\omega)\| = |c| \cdot \|\widetilde{M}^E(-n, \omega) v(\omega)\| \leq |c| \cdot D e^{-\kappa \cdot n}, \quad n \in \mathbb{N}$$

for some constants  $D, \kappa > 0$ . Consequently, the sequence  $(\alpha(n))_{n \in \mathbb{Z}}$  is an element of  $\ell^2(\mathbb{Z})$ . This implies that  $E \in \Sigma$  contradicting the fact that  $E$  was an element in the resolvent. Hence,  $u(\omega)$  and  $v(\omega)$  are linearly independent.

Assume that  $u(\omega)$  is not unique up to a sign. Then there are two linearly independent  $u^{(1)}(\omega), u^{(2)}(\omega) \in \mathbb{R}^2$  such that  $\|\widetilde{M}^E(n, \omega) u^{(1)}(\omega)\|$  and  $\|\widetilde{M}^E(n, \omega) u^{(2)}(\omega)\|$  tend to zero. For all  $x \in \mathbb{R}^2$  there are  $\lambda_1, \lambda_2 \in \mathbb{R}$  such that  $x = \lambda_1 \cdot u_1(\omega) + \lambda_2 \cdot u_2(\omega)$ . Thus,

$$\|\widetilde{M}^E(n, \omega) x\| \leq |\lambda_1| \cdot \underbrace{\|\widetilde{M}^E(n, \omega) u^{(1)}(\omega)\|}_{\rightarrow 0} + |\lambda_2| \cdot \underbrace{\|\widetilde{M}^E(n, \omega) u^{(2)}(\omega)\|}_{\rightarrow 0} \rightarrow 0, \quad n \rightarrow \infty$$

which contradicts the fact that  $\|\widetilde{M}^E(n, \omega)\| \geq 1$  for all  $n \in \mathbb{N}$ . This implies that  $u(\omega)$  is unique up to a sign and similarly  $v(\omega)$  is unique up to a sign. Denote by  $U(\omega)$  the corresponding unique one-dimensional subspace of  $\mathbb{R}^2$  generated by  $u(\omega)$  and analogously  $V(\omega)$  for  $v(\omega)$ .

Next, define a matrix  $C(\omega) := (u(\omega), v(\omega))$ . According to the previous considerations, the matrix  $C(\omega)$  is invertible. As mentioned in the beginning of Section 2, we know that  $\widetilde{M}^E(n, T\omega) \widetilde{M}^E(\omega) = \widetilde{M}^E(n+1, \omega)$ . Thus,  $\|\widetilde{M}^E(n, T\omega) x(T\omega)\|$  is exponentially decaying for the vector  $x(T\omega) := \widetilde{M}^E(\omega) u(\omega)$ . As we have seen above, there can be at most one one-dimensional subspace  $U(T\omega) \subsetneq \mathbb{R}^2$  such that the solutions decay exponentially for  $T\omega \in \Omega$ . Consequently,  $x(T\omega)$  is an element of  $U(T\omega)$  and so, there exists a  $d(\omega) \in \mathbb{R}$  such that  $\widetilde{M}^E(\omega) u(\omega) = d(\omega) \cdot u(T\omega)$ , where  $u(T\omega)$  is the unique vector (up to a sign) with norm one for  $T\omega \in \Omega$ . Analogously, there exists an  $e(\omega) \in \mathbb{R}$  such that  $\widetilde{M}^E(\omega) v(\omega) = e(\omega) \cdot v(T\omega)$ . Hence,

$$\begin{aligned} C^{-1}(T\omega) \widetilde{M}^E(\omega) C(\omega) &= \begin{pmatrix} C^{-1}(T\omega) \widetilde{M}^E(\omega) u(\omega) & C^{-1}(T\omega) \widetilde{M}^E(\omega) v(\omega) \end{pmatrix} \\ &= \begin{pmatrix} d(\omega) \cdot C^{-1}(T\omega) u(T\omega) & e(\omega) \cdot C^{-1}(T\omega) v(T\omega) \end{pmatrix} \\ &= \begin{pmatrix} d(\omega) & 0 \\ 0 & e(\omega) \end{pmatrix}. \end{aligned}$$

Multiplying  $u(\omega)$ ,  $v(\omega)$  or both with minus one will add a minus sign to  $d(\omega)$  respectively  $e(\omega)$ . Since, further,  $u(\omega)$  and  $v(\omega)$  can be chosen continuously in a neighborhood of  $\omega \in \Omega$ ,

by changing the sign, it follows that the maps  $\|C\|, \|C^{-1}\|, |d|, |e| : \Omega \rightarrow \mathbb{R}$  are continuous. The map  $\widetilde{M}^E(\omega)$  is invertible implying that  $d(\omega)$  and  $e(\omega)$  never vanish. Thus, the function  $\widetilde{M}^E$  is uniform by Lemma 3.6. According to Lemma 3.5, the continuous function  $M^E$  is uniform as well. ■

## 5. THE MAIN RESULTS

In our main Theorem 5.1, we give a complete description of the spectrum of Jacobi operators in terms of the Lyapunov exponent  $\gamma(E)$  and the uniformity properties of the transfer matrices  $M^E$ , ( $E \in \mathbb{R}$ ). As a special case, we show in Corollary 5.2 that uniformity of all the  $M^E$  is equivalent to the fact that the spectrum is exactly the set of zeros of the Lyapunov exponent  $\gamma(\cdot)$  as a function of  $E \in \mathbb{R}$ . Furthermore, the uniformity condition guarantees the continuity of the function  $\gamma(\cdot)$ .

**Theorem 5.1.** *Let  $(\Omega, T)$  be strictly ergodic and consider a family of Jacobi operators  $(H_\omega)_{\omega \in \Omega}$ . Then the spectrum  $\Sigma$  is equal to the disjoint union*

$$\{E \in \mathbb{R} \mid \gamma(E) = 0\} \bigsqcup \{E \in \mathbb{R} \mid M^E \text{ is not uniform}\}.$$

*Proof.* It follows from Lemma 4.8 that these sets are disjoint. It suffices to show that the equation

$$\Sigma^C = \{E \in \mathbb{R} \mid M^E \text{ is uniform and } \gamma(E) > 0\}$$

holds. This follows immediately by Proposition 4.4 and Lemma 4.10. ■

**Corollary 5.2.** *Let  $(\Omega, T)$  be strictly ergodic. Then the following assertions are equivalent.*

- (i) *The matrix  $M^E$  is uniform for each  $E \in \mathbb{R}$ .*
- (ii)  $\Sigma = \{E \in \mathbb{R} \mid \gamma(E) = 0\}$

*In this case,  $\gamma : \mathbb{R} \rightarrow [0, \infty)$  is continuous.*

*Proof.* This equivalence follows immediately by Theorem 5.1. The continuity of  $\gamma$  is a consequence of Lemma 4.7. ■

## 6. SUBSHIFTS

We will now apply our previous results to the special case of a dynamical system induced by a subshift. To do so, we first recapitulate some well-known results about the spectrum of the family of Jacobi operators corresponding to a subshift. Then our main statement will be that the spectrum of a Jacobi operator, generated by an aperiodic subshift, with some reasonable requirements is a Cantor set of Lebesgue measure zero, if the transfer matrix  $M^E$  is uniform for each  $E \in \mathbb{R}$  (Theorem 6.4). The main idea of the proof is to apply an assertion of [Rem11]. Further, we recall the notion of the Boshernitzan condition for subshifts. Indeed, a large class of subshifts satisfies this condition. It turns out that in this case the transfer matrices are uniform for all energies.

Consider a finite alphabet  $\mathcal{A} \subsetneq \mathbb{R}$  and  $\mathcal{A}^{\mathbb{Z}} := \{\varphi : \mathbb{Z} \rightarrow \mathcal{A}\}$ . Denote by  $d_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \rightarrow \{0, 1\}$  the discrete metric on  $\mathcal{A}$ . We define a metric  $d : \mathcal{A}^{\mathbb{Z}} \times \mathcal{A}^{\mathbb{Z}} \rightarrow [0, \infty)$  on  $\mathcal{A}^{\mathbb{Z}}$  by

$$d(\varphi, \psi) := \sum_{k=-\infty}^{\infty} \frac{d_{\mathcal{A}}(\varphi(k), \psi(k))}{2^{|k|}}.$$

Then a well-known result is that  $(\mathcal{A}^{\mathbb{Z}}, d)$  is compact, see [Wal82].

Let  $(\Omega, T)$  be a subshift over  $\mathcal{A}$ , where  $\Omega$  is a closed subset of  $\mathcal{A}^{\mathbb{Z}}$  and invariant under the homeomorphism  $T : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$  defined by

$$(T\omega)(n) := \omega(n+1), \quad \omega \in \Omega.$$

This map is also called the *shift operator*. For each  $\omega \in \Omega$  we have the set of words associated to  $\omega$  given by

$$\mathcal{W}_\omega := \{\omega(l) \dots \omega(l+n-1) \mid l \in \mathbb{Z}, n \in \mathbb{N}\}.$$

Further,  $\mathcal{W}(\Omega) := \bigcup_{\omega \in \Omega} \mathcal{W}_\omega$  is the set of words associated to  $\Omega$ . We say a subshift  $(\Omega, T)$  is *aperiodic*, if for all  $\omega \in \Omega$  there is no  $0 \neq m \in \mathbb{N}$  such that  $T^m \omega = \omega$ . If for  $\omega \in \Omega$  there exists a  $0 \neq m \in \mathbb{N}$  such that  $T^m \omega = \omega$  this element is called *periodic* and  $m$  is the *period* of  $\omega$ .

First of all we recapitulate some well-known results, see e.g. the textbooks [Tes00, CL90].

**Proposition 6.1.** *Let  $(\Omega, T)$  be a uniquely ergodic dynamical system induced by a subshift. Then there are  $\Sigma_{ac}, \Sigma_{sc}, \Sigma_{pp} \subset \mathbb{R}$  such that*

$$\begin{aligned} \Sigma_{ac} &= \sigma_{ac}(H_\omega) \quad a.s., \\ \Sigma_{sc} &= \sigma_{sc}(H_\omega) \quad a.s., \\ \Sigma_{pp} &= \sigma_{pp}(H_\omega) \quad a.s.. \end{aligned}$$

Further, for  $\mu$ -almost every  $\omega \in \Omega$  the set  $\sigma(H_\omega)$  has no discrete points.

Now the following assertion immediately follows by Proposition 4.3.

**Corollary 6.2.** *Let  $(\Omega, T)$  be a strictly ergodic dynamical system induced by a subshift. Then for every  $\omega \in \Omega$  the set  $\sigma(H_\omega)$  does not contain a discrete point.*

*Remark.* If  $(\Omega, T)$  is minimal, a stronger result can be shown, namely that  $\Sigma_{ac} = \sigma_{ac}(H_\omega)$  for all  $\omega \in \Omega$ , see [LS99], Theorem 6.1.

Consider for a subshift  $(\Omega, T)$  the restriction  $(\Omega^+, T^+)$  respectively  $(\Omega^-, T^-)$  defined as follows:

$$\begin{aligned} \Omega^+ &:= \{\omega|_{\mathbb{N}_0} \mid \omega \in \Omega\} \text{ with } T^+ := T, \\ \Omega^- &:= \{\omega|_{\mathbb{Z} \setminus \mathbb{N}_0} \mid \omega \in \Omega\} \text{ with } T^- := T^{-1}. \end{aligned}$$

Recall for  $\omega \in \Omega$  the definition of the Jacobi operator  $H_\omega$  with the corresponding continuous maps  $p : \Omega \rightarrow \mathbb{R} \setminus \{0\}$  and  $q : \Omega \rightarrow \mathbb{R}$ . For  $\omega^+ \in \Omega^+$  denote by  $H_{\omega^+}$  the restriction of the Jacobi operator  $H_\omega$  where  $\omega^+ = \omega|_{\mathbb{N}_0}$ . Similarly, denote the corresponding restrictions of  $p$  and  $q$  by  $p^+$  and  $q^+$ , which are still continuous.

A sequence  $u := (u(n))_{n \in \mathbb{N}_0}$  is called *eventually periodic* if there exist  $K, m \in \mathbb{N}$  such that  $u$  is periodic outside of  $\{0, 1, \dots, K\}$  with period  $m$ . For an  $\omega^+ \in \Omega^+$  eventual periodicity is defined accordingly by considering the sequence  $((T^+)^n \omega^+)_{n \in \mathbb{N}_0}$ . A closed set is called a *Cantor set*, if it does not contain a non-trivial interval and no discrete points. Note that a set of Lebesgue measure zero cannot have a non-trivial interval at all.

**Lemma 6.3.** *Let  $(\Omega, T)$  be an aperiodic subshift. Then  $(\Omega^+, T^+)$  and  $(\Omega^-, T^-)$  do not contain an eventually periodic element.*

*Proof.* We show that  $(\Omega^+, T^+)$  does not have an eventually periodic element. The proof for  $(\Omega^-, T^-)$  works similarly. Assume that there exists an eventually periodic element  $\omega^+$  with period  $m \in \mathbb{N}$ . Then there is an  $\omega \in \Omega$  such that  $\omega^+ = \omega|_{\mathbb{N}_0}$ . Consider the sequence  $\omega_n := T^{-n \cdot m} \omega$ . Then by compactness of  $\Omega$  there is a convergent subsequence  $\omega_{n_j}$  converging to some  $z \in \Omega$ . Since  $\omega^+$  is eventually periodic, it follows that  $z$  is periodic, contradicting the fact that  $(\Omega, T)$  is aperiodic.  $\blacksquare$

Consider the dynamical system  $(\tilde{\Omega}, \tilde{T})$  defined by

$$\tilde{\Omega} := \left\{ \tilde{\omega} := \begin{pmatrix} p(\omega) \\ q(\omega) \end{pmatrix} \mid \omega \in \Omega \right\}$$

endowed with the product topology and

$$\tilde{T}\tilde{\omega} := \begin{pmatrix} p(T\omega) \\ q(T\omega) \end{pmatrix}, \quad \tilde{\omega} \in \tilde{\Omega}.$$

Note that the notion of periodicity and eventual periodicity carries over for  $\omega \in \Omega$  to  $p(\omega)$  and  $q(\omega)$  in the obvious way. Precisely,  $p(\omega)$  is periodic, if there exists a  $0 \neq m \in \mathbb{N}$  such that  $p(T^m \omega) = p(\omega)$ .

If  $(\Omega, T)$  is minimal, the dynamical system  $(\tilde{\Omega}, \tilde{T})$  is minimal as well. Our aim is to show that the spectrum of a family of Jacobi operators is supported on a Cantor set of Lebesgue measure zero, if  $p(\omega)$  or  $q(\omega)$  is not periodic for each  $\omega \in \Omega$ . In this section, we consider Jacobi operators  $H_\omega$  associated with an aperiodic subshift such that the aperiodicity of the subshifts carries over to  $(\tilde{\Omega}, \tilde{T})$ . Denote this condition by (A). This is for example the case, if  $p$  or  $q$  is injective.

**Theorem 6.4.** *Let  $(\Omega, T)$  be a strictly ergodic and aperiodic subshift. Consider the continuous maps  $p : \Omega \rightarrow \mathbb{R} \setminus \{0\}$  and  $q : \Omega \rightarrow \mathbb{R}$  which take finitely many values with corresponding family of Jacobi operators  $(H_\omega)_{\omega \in \Omega}$ . Suppose that condition (A) is satisfied and that the transfer matrix  $M^E$  is uniform for every  $E \in \mathbb{R}$ . Then the spectrum  $\Sigma$  is a Cantor set of Lebesgue measure zero.*

*Proof.* Since  $\Sigma$  does not contain discrete points, we have to verify that  $|\Sigma| = 0$  where  $|\cdot|$  denotes the Lebesgue measure, see Corollary 6.2. By Corollary 5.2 it is enough to verify that  $|\Gamma| = 0$ . According to general results, (see e.g. [Tes00], Theorem 5.17) it is sufficient to show that  $\Sigma_{ac}$  is empty.

Let  $\omega \in \Omega$  be chosen such that  $\sigma_{ac}(H_\omega) = \Sigma_{ac}$ . Assume that  $\sigma_{ac}(H_\omega)$  is non-empty. Since any perturbation of finite range does not change the absolutely continuous spectrum it follows that  $\sigma_{ac}(H_\omega^+)$  or  $\sigma_{ac}(H_\omega^-)$  is non-empty. Without loss of generality, let  $\sigma_{ac}(H_\omega^+) \neq \emptyset$  and so,  $p^+(T^{(\cdot)}\omega)$  and  $q^+(T^{(\cdot)}\omega)$  are eventually periodic, see [Rem11], Theorem 1.1. Hence, there is an eventually periodic element of the dynamical system  $(\tilde{\Omega}^+, \tilde{T}^+)$ . However, this contradicts the assertion of Lemma 6.3.  $\blacksquare$

In the work [DL06a] it is shown that the so called Boshernitzan condition, first introduced in [Bos85], implies for a minimal subshift that a locally constant (definition see below), continuous function  $A : \Omega \rightarrow SL(2, \mathbb{R})$  is uniform. Indeed, a large class of subshifts satisfies this condition. For instance, in the work [DL06a] it is shown that subshifts obeying positive weights (PW) satisfies the Boshernitzan condition. The class of linear repetitive (or linear recurrent) subshifts is contained in the class of subshifts with (PW). Actually, it turns out

that these two classes are equal by unpublished results of Boshernitzan, see [BBL13] as well. Also all Sturmian subshifts fulfill the Boshernitzan condition. Further, almost all interval exchange transformations, almost all circle maps and almost all Arnoux-Rauzy subshifts satisfy the Boshernitzan condition, see [DL06b].

A continuous function  $M$  on  $\Omega$  is called *locally constant*, if there exists an  $N \in \mathbb{N}$  such that for  $\omega_1, \omega_2 \in \Omega$  whenever  $(\omega_1(-N), \dots, \omega_1(N)) = (\omega_2(-N), \dots, \omega_2(N))$ , then the equality  $M(\omega_1) = M(\omega_2)$  holds.

Let  $(\Omega, T)$  be a subshift over the finite alphabet  $\mathcal{A} \subsetneq \mathbb{R}$  and  $\mathcal{W}(\Omega)$  be the set of the associated words of  $\Omega$ . For  $v \in \Omega$  we define the subset of all elements of  $\Omega$  which begin with the associated word  $v$  by

$$\mathcal{V}_v := \{\omega \in \Omega \mid \omega(1) \cdots \omega(|v|) = v\}.$$

Note that  $|v|$  denotes the length of the associated word  $v \in \mathcal{W}(\Omega)$ . Further, for a  $T$ -invariant probability measure  $\mu$  on  $\Omega$  and  $n \in \mathbb{N}$  we define the number

$$\eta_\mu(n) := \min\{\mu(\mathcal{V}_v) \mid v \in \mathcal{W}(\Omega), |v| = n\}.$$

The following definition was originally introduced by Boshernitzan in his work [Bos85]. A subshift  $(\Omega, T)$  over the finite alphabet  $\mathcal{A} \subsetneq \mathbb{R}$  satisfies the *Boshernitzan condition*, if there exists an ergodic probability measure  $\mu$  on  $\Omega$  such that

$$\limsup_{n \rightarrow \infty} n \cdot \eta_\mu(n) > 0.$$

As mentioned before, a large class of subshifts fulfills this condition and the following statement, proven in [DL06a], gives us a useful tool.

**Theorem 6.5.** *Let  $(\Omega, T)$  be a minimal subshift over the finite alphabet  $\mathcal{A} \subsetneq \mathbb{R}$  satisfying the Boshernitzan condition. Then a locally constant map  $M : \Omega \rightarrow SL(2, \mathbb{R})$  is uniform.*

Theorem 6.5 provides a sufficient condition for the uniformity of all transfer matrices corresponding to a Jacobi operator and its subshift. Combining this with Theorem 6.4 we get the following assertion.

**Corollary 6.6.** *Let  $(\Omega, T)$  be a minimal, aperiodic subshift such that the Boshernitzan condition holds. Consider the family of the corresponding Jacobi operators  $\{H_\omega\}_{\omega \in \Omega}$  where the continuous maps  $p$  and  $q$  take finitely many values and they obey condition (A). Then the transfer matrix  $M^E : \Omega \rightarrow GL(2, \mathbb{R})$  is uniform for each  $E \in \mathbb{R}$ . In particular, the spectrum  $\Sigma$  is a Cantor set of Lebesgue measure zero.*

*Proof.* It is shown in [Bos92] (Theorem 1.2) that the Boshernitzan condition for a minimal subshift implies unique ergodicity of the subshift. Since  $p$  and  $q$  are uniformly continuous and only take finitely many values, it follows that they are locally constant. Thus, for  $E \in \mathbb{R}$  the continuous map  $\widetilde{M}^E : \Omega \rightarrow SL(2, \mathbb{R})$  is locally constant as well. According to Theorem 6.5 it follows that  $\widetilde{M}^E$  is uniform for each  $E \in \mathbb{R}$  and so is  $M^E$  as well, see Lemma 4.1. Consequently, we can apply Theorem 6.4 leading to our assertion. ■

#### ACKNOWLEDGMENT

We would like to express our thanks to our advisor DANIEL LENZ for his careful guidance and support. Especially, we thank him for coming up with the issue of this work and for fruitful suggestions concerning the preparation of the paper. It is our great pleasure to acknowledge



MATTHIAS KELLER for enlightening discussions about the connection of spectral aspects and subexponentially growing functions. FP would like to thank the German National Academic Foundation (Studienstiftung des deutschen Volkes) for financial support.

## REFERENCES

- [And58] Philip Warren Anderson, *Absence of diffusion in certain random lattices*, Phys. Rev. **109** (1958), 1492–1505.
- [BBG91] Jean Bellissard, Anton Bovier, and Jean-Michel Ghez, *Spectral properties of a tight binding Hamiltonian with period doubling potential*, vol. 135, 1991, pp. 379–399.
- [BBL13] Adnene Besbes, Michael Boshernitzan, and Daniel Lenz, *Delone Sets with Finite Local Complexity: Linear Repetitivity Versus Positivity of Weights*, Discrete Comput. Geom. **49** (2013), no. 2, 335–347.
- [BG93] Anton Bovier and Jean-Michel Ghez, *Spectral properties of one-dimensional Schrödinger operators with potentials generated by substitutions*, Comm. Math. Phys. **158** (1993), no. 1, 45–66.
- [BIST89] J. Bellissard, B. Iochum, E. Scoppola, and D. Testard, *Spectral properties of one-dimensional quasicrystals*, Comm. Math. Phys. **125** (1989), no. 3, 527–543.
- [Bos92] Michael D. Boshernitzan, *A condition for unique ergodicity of minimal symbolic flows*, Ergodic Theory Dynam. Systems **12** (1992), no. 3, 425–428.
- [Bos85] Michael Boshernitzan, *A unique ergodicity of minimal symbolic flows with linear block growth*, J. Analyse Math. **44** (1984/85), 77–96.
- [CKM87] René Carmona, Abel Klein, and Fabio Martinelli, *Anderson localization for Bernoulli and other singular potentials*, Comm. Math. Phys. **108** (1987), no. 1, 41–66.
- [CL90] René Carmona and Jean Lacroix, *Spectral theory of random Schrödinger operators*, Birkhäuser Boston Inc., 1990.
- [Dah10] Janine M. Dahl, *The Spectrum of the Off-diagonal Fibonacci Operator*, Ph.D. thesis, Rice University, 2010, p. 95.
- [Dam98] David Damanik, *Singular continuous spectrum for a class of substitution Hamiltonians*, Lett. Math. Phys. **46** (1998), no. 4, 303–311.
- [DG08] David Damanik and Anton Gorodetski, *The Spectrum and the Spectral type of the Off-diagonal Fibonacci Operator*, arXiv:0807.3024 (2008).
- [DKL00] David Damanik, Rowan Killip, and Daniel Lenz, *Uniform spectral properties of one-dimensional quasicrystals. III.  $\alpha$ -continuity*, Comm. Math. Phys. **212** (2000), no. 1, 191–204.
- [DL99a] David Damanik and Daniel Lenz, *Uniform spectral properties of one-dimensional quasicrystals. I. Absence of eigenvalues*, Comm. Math. Phys. **207** (1999), no. 3, 687–696.
- [DL99b] ———, *Uniform spectral properties of one-dimensional quasicrystals. II. The Lyapunov exponent*, Lett. Math. Phys. **50** (1999), no. 4, 245–257.
- [DL06a] ———, *A condition of Boshernitzan and uniform convergence in the multiplicative ergodic theorem*, Duke Math. J. **133** (2006), no. 1, 95–123.
- [DL06b] ———, *Zero-measure Cantor spectrum for Schrödinger operators with low-complexity potentials*, J. Math. Pures Appl. (9) **85** (2006), no. 5, 671–686.
- [EFHN09] Tatjana Eisner, Balint Farkas, Markus Haase, and Rainer Nagel, *Ergodic theory - an operatortheoretic approach*, Manuscript 12th International Internet Seminar, <http://isem.mathematik.tu-darmstadt.de/isem>, 18.01.2013, 2009.
- [FMSS85] J. Fröhlich, F. Martinelli, E. Scoppola, and T. Spencer, *Constructive proof of localization in the Anderson tight binding model*, Comm. Math. Phys. **101** (1985), no. 1, 21–46.
- [Fur97] Alex Furman, *On the multiplicative ergodic theorem for uniquely ergodic systems*, Ann. Inst. H. Poincaré Probab. Statist. **33** (1997), no. 6, 797–815.
- [JNS09] Jan Janas, Serguei Naboko, and Günter Stolz, *Decay bounds on eigenfunctions and the singular spectrum of unbounded Jacobi matrices*, Int. Math. Res. Not. IMRN (2009), no. 4, 736–764.
- [Kir08] Werner Kirsch, *An invitation to random Schrödinger operators*, Random Schrödinger operators, Panor. Synthèses, vol. 25, Soc. Math. France, 2008, With an appendix by Frédéric Klopp, pp. 1–119.
- [KKT83] Mahito Kohmoto, Leo P. Kadanoff, and Chao Tang, *Localization problem in one dimension: mapping and escape*, Phys. Rev. Lett. **50** (1983), no. 23, 1870–1872.

- [Kot89] Shinichi Kotani, *Jacobi matrices with random potentials taking finitely many values*, Rev. Math. Phys. **1** (1989), no. 1, 129–133.
- [KW82] Yitzhak Katznelson and Benjamin Weiss, *A simple proof of some ergodic theorems*, Israel J. Math. **42** (1982), no. 4, 291–296.
- [Len99] Daniel Lenz, *Random operators and crossed products*, Math. Phys. Anal. Geom. **2** (1999), no. 2, 197–220.
- [Len02] ———, *Singular spectrum of Lebesgue measure zero for one-dimensional quasicrystals*, Comm. Math. Phys. **227** (2002), no. 1, 119–130.
- [Len04] ———, *Existence of non-uniform cocycles on uniquely ergodic systems*, Ann. Inst. H. Poincaré Probab. Statist. **40** (2004), no. 2, 197–206.
- [LS99] Yoram Last and Barry Simon, *Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators*, Invent. Math. **135** (1999), no. 2, 329–367.
- [Mar12] Laurent Marin, *On- and off-diagonal Sturmian operators: dynamic and spectral dimension*, Rev. Math. Phys. **24** (2012), no. 5, 1250011, 23.
- [OPR<sup>+</sup>83] Stellan Ostlund, Rahul Pandit, David Rand, Hans Joachim Schellnhuber, and Eric D. Siggia, *One-dimensional schrödinger equation with an almost periodic potential*, Phys. Rev. Lett. **50** (1983), 1873–1876.
- [Rem11] Christian Remling, *The absolutely continuous spectrum of Jacobi matrices*, Ann. of Math. (2) **174** (2011), no. 1, 125–171.
- [SBGC84] D. Shechtman, I. Blech, D. Gratias, and J. W. Cahn, *Metallic phase with long-range orientational order and no translational symmetry*, Phys. Rev. Lett. **53** (1984), 1951–1953.
- [Süt87] András Sütő, *The spectrum of a quasiperiodic Schrödinger operator*, Comm. Math. Phys. **111** (1987), no. 3, 409–415.
- [Süt89] ———, *Singular continuous spectrum on a Cantor set of zero Lebesgue measure for the Fibonacci Hamiltonian*, J. Statist. Phys. **56** (1989), no. 3-4, 525–531.
- [Tes00] Gerald Teschl, *Jacobi operators and completely integrable nonlinear lattices*, vol. 72, American Mathematical Society, 2000.
- [Wal82] Peter Walters, *An introduction to ergodic theory*, vol. 79, Springer-Verlag, 1982.
- [Yes12] William N. Yessen, *On the spectrum of 1d quantum ising quasicrystal*, arXiv:1110.6894 (2012).

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